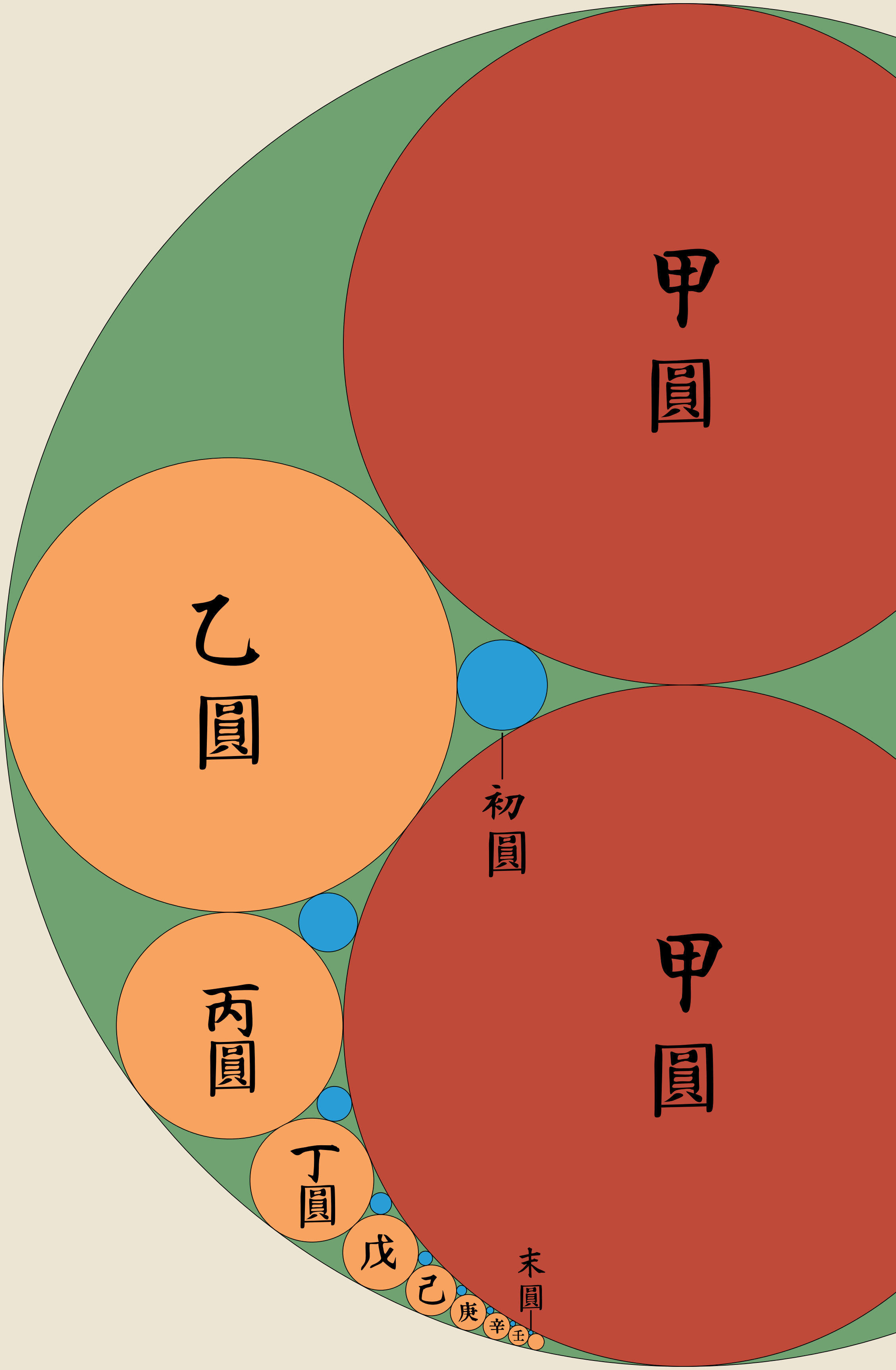
所懸于東都本所栁島妙見堂者一事



Japanese Temple Geometry

During Japan's period of national seclusion (1639–1854), native mathematics thrived, as evidenced in sangaku*—wooden tablets engraved with geometry problems hung under the roofs of shrines and temples*

by Tony Rothman, with the cooperation of Hidetoshi Fukagawa

Of the world's countless customs and traditions, perhaps none is as elegant, nor as beautiful, as the tradition of *sangaku,* Japanese temple geometry. From 1639 to 1854, Japan lived in strict, self-imposed isolation from the West. Access to all forms of occidental culture was suppressed, and the influx of Western scientific ideas was effectively curtailed. During this period of seclusion, a kind of native mathematics flourished.

Devotees of math, evidently samurai, merchants and farmers, would solve a wide variety of geometry problems, inscribe their efforts in delicately colored wooden tablets and hang the works under the roofs of religious buildings. These *sangaku,* a word that literally means mathematical tablet, may have been acts of homage—a thanks to a guiding spirit—or they may have been brazen challenges to other worshipers: Solve this one if you can!

For the most part, *sangaku* deal with ordinary Euclidean geometry. But the problems are strikingly different from those found in a typical high school geometry course. Circles and ellipses play a far more prominent role than in Western problems: circles within ellipses, ellipses within circles. Some of the exercises are quite simple and could be solved by first-year students. Others are nearly impossible, and modern geometers invariably tackle them with advanced methods, including calculus and affine transformations. Although most of the problems would be classified today as recreational or educational mathematics, a few predate known Western results, such as the Malfatti theorem, the Casey theorem and the Soddy hexlet theorem. One problem reproduces the Descartes circle theorem. Many of the tablets are exceptionally beautiful and can be regarded as works of art.

Pleasing the *Kami*

It is natural to wonder who created the *sangaku* and when, but it is easier to ask such questions than to answer them. The custom of hanging tablets at shrines was established in Japan centuries before *sangaku* came into existence. Shintoism, Japan's native religion, is populated by "eight hundred myriads of gods," the *kami.* Because the *kami,* it was said, love horses, those worshipers who could not present a living horse as an offering to the shrine might instead give a likeness drawn on wood. As a result, many tablets dating from the 15th century and earlier depict horses.

Of the *sangaku* themselves, the oldest surviving tablet has been found in Tochigi Prefecture and dates from 1683. Another tablet, from Kyoto, is dated 1686, and a third is from 1691. The 19th-century travel diary of the mathematician Kazu Yamaguchi refers to an even earlier tablet—now lost—dated 1668. So historians guess that the custom first arose in the second half of the 17th century. In 1789 the first collection containing typical *sangaku* problems was published. Other collections followed throughout the 18th and 19th centuries. These books were either handwritten or printed with wooden blocks and are remarkably beautiful. Today more than 880 tablets survive, with references to hundreds of others in the various collections. From a survey of the extant *sangaku,* the tablets seem to have been distributed fairly uniformly throughout Japan, in both rural and urban districts, with about twice as many found in Shinto shrines as in Buddhist temples.

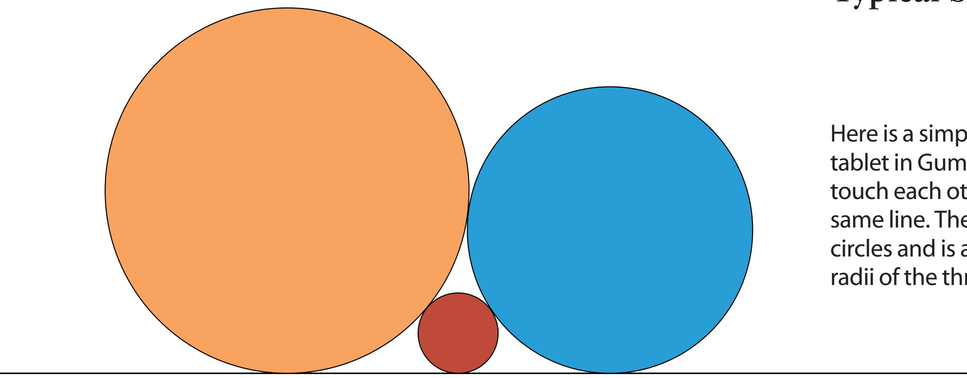
Most of the surviving *sangaku* contain more than one theorem and are frequently brightly colored. The proof of the theorem is usually not given, only the result. Other information typically includes the name of the presenter and the date. Not all the problems deal solely with geometry. Some ask for the volumes of various solids and thus require calculus. (This point raises the interesting question of what techniques the practitioners brought into play; some speculations will be offered in the following discussion.) Other tablets contain Diophantine problems—that is, algebraic equations requiring solutions in integers.

In modern times the *sangaku* have been largely forgotten but for a few devotees of traditional Japanese mathematics. Among them is Hidetoshi Fukagawa, a high school teacher in Aichi Prefecture, roughly halfway between Tokyo and Osaka. About 30 years ago Fukagawa decided to study the history of Japanese mathematics in hopes of finding better ways to teach his courses. A mention of the math tablets in an old library book greatly astonished him, for he had never heard of such a thing. Since then, Fukagawa, who holds a Ph.D. in mathematics, has traveled widely in Japan to study the tablets and has amassed a collection of books dealing not only

SANGAKU PROBLEMS typically involve multitudes of circles within circles or of spheres within other figures. This problem is from a *sangaku,* or mathematical wooden tablet, dated 1788 in Tokyo Prefecture. It asks for the radius of the nth largest blue circle in terms of r, the radius of the green circle. Note that the red circles are identical, each with radius $r/2$. (Hint: The radius of the fifth blue circle is $r/95$.) The original solution to this problem deploys the Japanese equivalent of the Descartes circle theorem. (The answer can be found on page 91.)

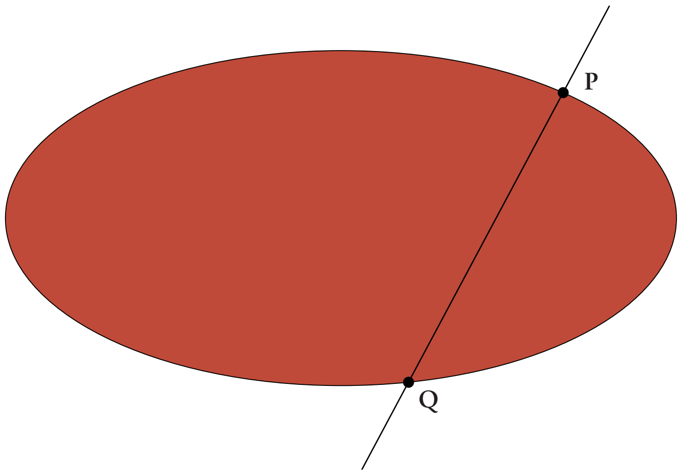
BRYAN CHRISTIE, AFTER TOSHIHISA IWASAKI

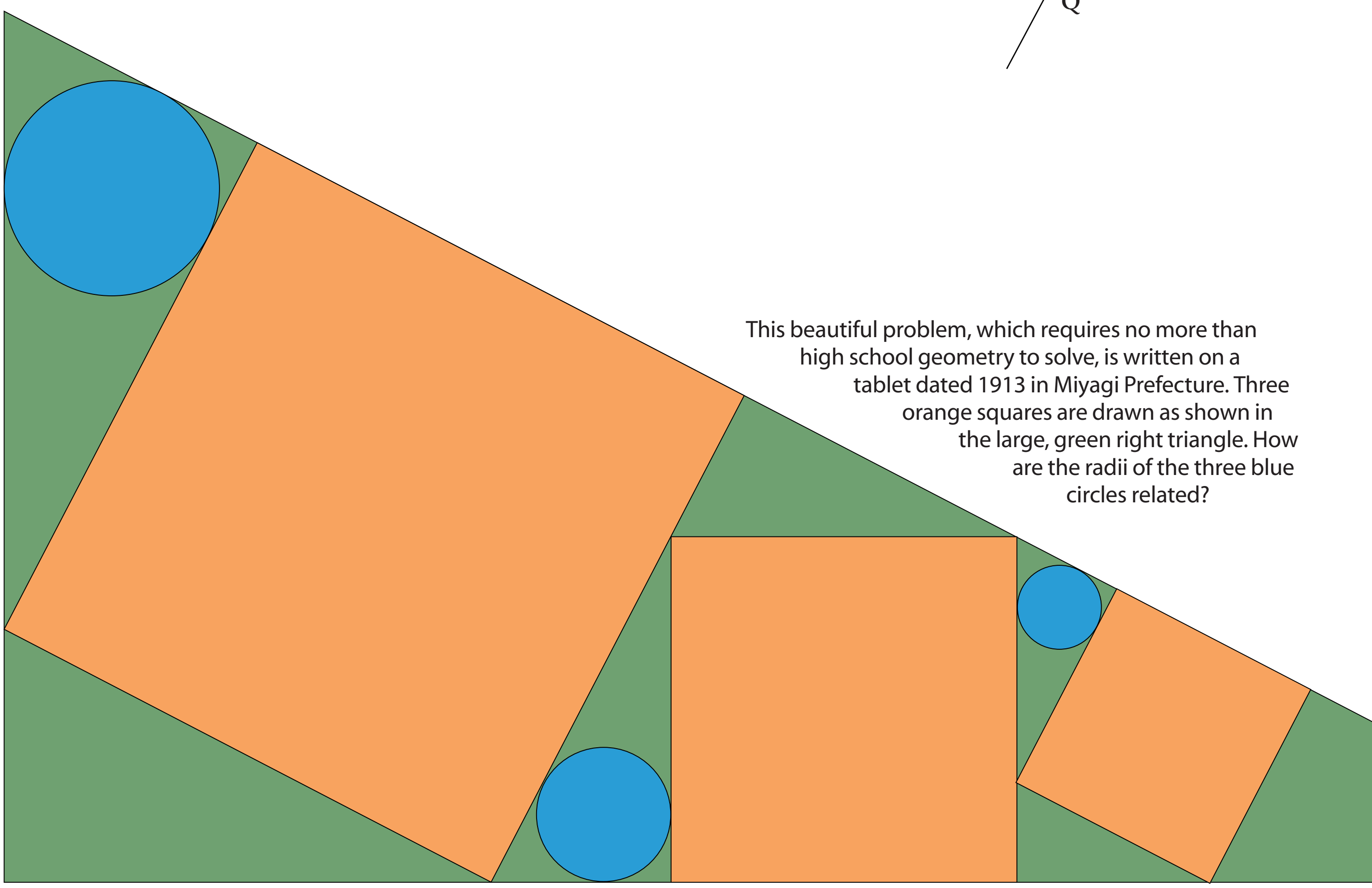
Typical *Sangaku* Problems*



Here is a simple problem that has survived on an 1824 tablet in Gumma Prefecture. The orange and blue circles touch each other at one point and are tangent to the same line. The small red circle touches both of the larger circles and is also tangent to the same line. How are the radii of the three circles related?

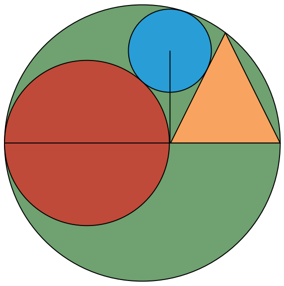
This striking problem was written in 1912 on a tablet extant in Miyagi Prefecture; the date of the problem itself is unknown. At a point P on an ellipse, draw the normal PQ such that it intersects the other side. Find the least value of PQ. At first glance, the problem appears to be trivial: the minimum PQ is the minor axis of the ellipse. Indeed, this is the solution if $b < a \leq \sqrt{2}b$, where a and b are the major and minor axes, respectively; however, the tablet does not give this solution but another, if $2b^2 < a^2$.



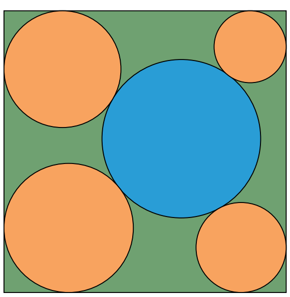


This beautiful problem, which requires no more than high school geometry to solve, is written on a tablet dated 1913 in Miyagi Prefecture. Three orange squares are drawn as shown in the large, green right triangle. How are the radii of the three blue circles related?

BRYAN CHRISTIE



In this problem, from an 1803 *sangaku* found in Gumma Prefecture, the base of an isosceles triangle sits on a diameter of the large green circle. This diameter also bisects the red circle, which is inscribed so that it just touches the inside of the green circle and one vertex of the triangle, as shown. The blue circle is inscribed so that it touches the outsides of both the red circle and the triangle, as well as the inside of the green circle. A line segment connects the center of the blue circle and the intersection point between the red circle and the triangle. Show that this line segment is perpendicular to the drawn diameter of the green circle.



This problem comes from an 1874 tablet in Gumma Prefecture. A large blue circle lies within a square. Four smaller orange circles, each with a different radius, touch the blue circle as well as the adjacent sides of the square. What is the relation between the radii of the four small circles and the length of the side of the square? (Hint: The problem can be solved by applying the Casey theorem, which describes the relation between four circles that are tangent to a fifth circle or to a straight line.)

**Answers are on page 91.*

with *sangaku* but with the general field of traditional Japanese mathematics.

To carry out his research, Fukagawa had to teach himself *Kambun*, an archaic form of Japanese that is closely related to Chinese. *Kambun* is the Japanese equivalent of Latin; during the Edo period (1603–1867), scientific works were written in this language, and only a few people in modern Japan are able to read it fluently. As new tablets have been discovered, Fukagawa has been called in to decipher them. In 1989 Fukagawa, along with Daniel Pedoe, published the first collection of *sangaku* in English. Most of the geometry problems accompanying this article were drawn from that collection.

Wasan versus *Yosan*

Although the origin of the *sangaku* cannot be pinpointed, it can be localized. There is a word in Japanese, *wasan,* that is used to refer to native Japanese mathematics. *Wasan* is meant to stand in opposition to *yosan,* or Western mathematics. To understand how *wasan* came into existence—and with it the unusual *sangaku* problems—one must first appreciate the peculiar history of Japanese mathematics.

Of the earliest times, very little is definitely known about mathematics in Japan, except that a system of exponential notation, similar to that employed by Archimedes in the *Sand Reckoner,* had been developed. More concrete information dates only from the mid-sixth century A.D., when Buddhism—and, with it, Chinese mathematics—made its way to Japan. Judging from the works that were taught at official schools at the start of the eighth century, historians infer that Japan had imported the great Chinese classics on arithmetic, algebra and geometry.

According to tradition, the earliest of these is the *Chou-pei Suan-ching,* which contains an example of the Pythagorean theorem and the diagram commonly used to prove it. This part of the tome is at least as old as the sixth century B.C.

A more advanced state of knowledge is represented in the *Chiu-chang Suan-shu,* considered the most influential of Chinese books on mathematics. The *Chiu-chang* describes methods for finding the areas of triangles, quadrilaterals, circles and other figures. It also contains simple word problems of the type that torment many high school students today: "If five oxen and two sheep cost eight taels of gold, and two oxen and eight sheep cost eight taels, what is the price of each animal?" The dates of the *Chiu-chang* are also uncertain, but most of it was probably composed by the third century B.C. If this information is correct, the *Chiu-chang* contains perhaps the first known mention of negative numbers and an early statement of the quadratic equation. (According to some historians, the ancient Egyptians had begun studying quadratic equations centuries before, prior to 2000 B.C.)

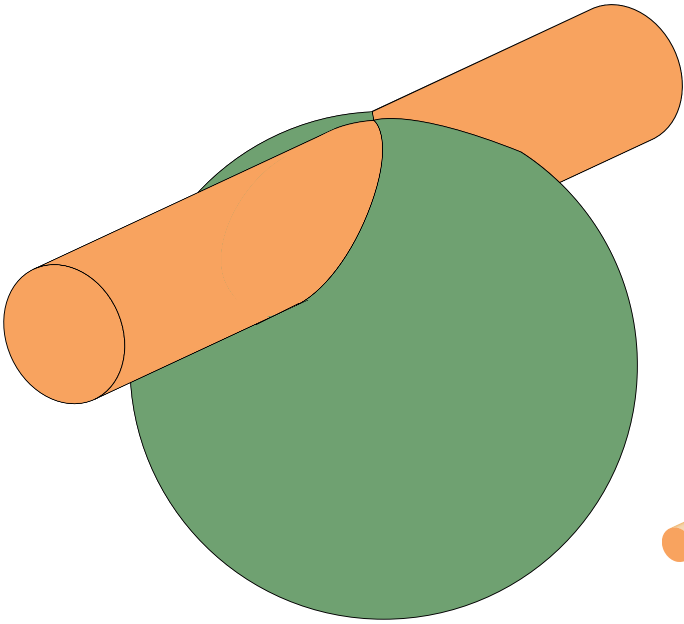
Despite the influx of Chinese learning, mathematics did not then take root in Japan. Instead the country entered a dark age, roughly contemporaneous with that of Western Europe. In the West, church and monastery became the centers of learning; in Japan, Buddhist temples served the same function, although mathematics does not seem to have played much of a role. By some accounts, during the Ashikaga shogunate (1338–1573) there could hardly be found in all Japan a person versed in the art of division.

It is not until the opening of the 17th century that definite historical records exist of any Japanese mathematicians. The first of these is Kambei Mori, who prospered around the year 1600. Although only one of Mori's works—a booklet—survives, he is known to have been instrumental in developing arithmetical calculations on the *soroban,* or Japanese abacus, and in popularizing it throughout the country.

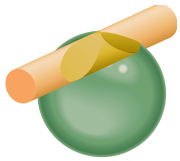
The oldest substantial Japanese work on mathematics actually extant belongs to Mori's pupil Koyu Yoshida (1598–1672). The book, entitled *Jinko-ki* (literally, "small and large numbers"), was published in 1627 and also concerns operations on the *soroban. Jinko-ki* was so influential that the name of the work often was synonymous with arithmetic. Because of the book's influence, computation—as opposed to logic—became the most important concept in traditional Japanese mathematics. To the extent that it makes sense to credit anyone with the founding of *wasan,* that honor probably goes to Mori and Yoshida.

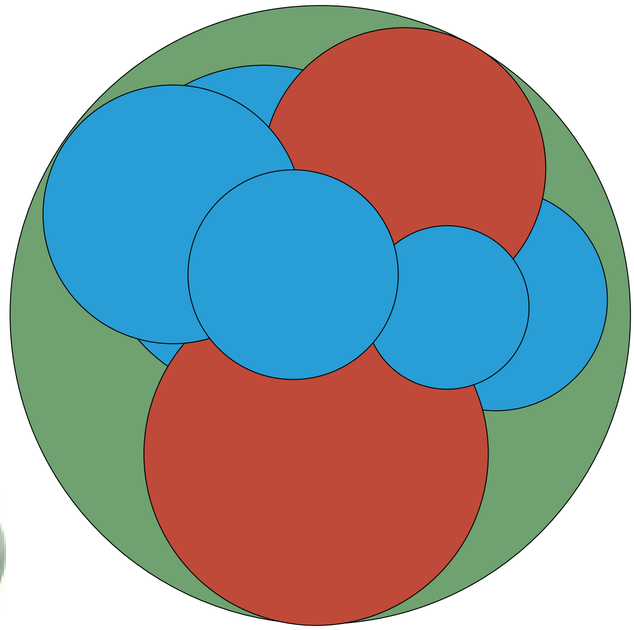
A Brilliant Flowering

Wasan, though, was created not so much by a few individuals but by something much larger. In 1639 the ruling Tokugawa shogunate (during the Edo period), to strengthen its power and diminish challenges to its reign, de-

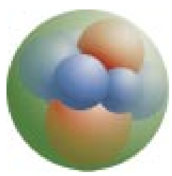


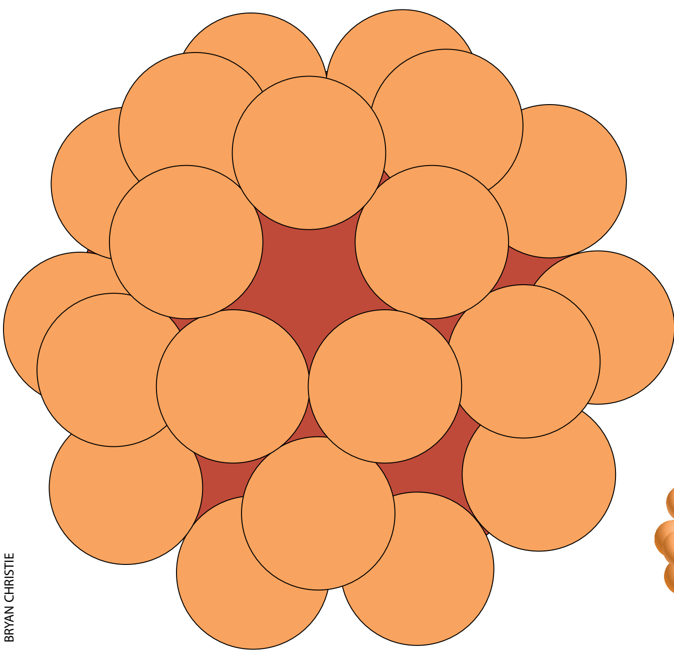
From a *sangaku* dated 1825, this problem was probably solved by using the *enri*, or the Japanese circle principle. A cylinder intersects a sphere so that the outside of the cylinder is tangent to the inside of the sphere. What is the surface area of the part of the cylinder contained inside the sphere? (The inset shows a three-dimensional view of the problem.)



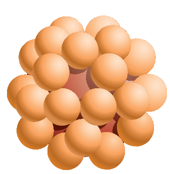


This problem is from an 1822 tablet in Kanagawa Prefecture. It predates by more than a century a theorem of Frederick Soddy, the famous British chemist who, along with Ernest Rutherford, discovered transmutation of the elements. Two red spheres touch each other and also touch the inside of the large green sphere. A loop of smaller, different-size blue spheres circle the "neck" between the red spheres. Each blue sphere in the "necklace" touches its nearest neighbors, and they all touch both the red spheres and the green sphere. How many blue spheres must there be? Also, how are the radii of the blue spheres related? (The inset shows a three-dimensional view of the problem.)





Hidetoshi Fukagawa was so fascinated with this problem, which dates from 1798, that he built a wooden model of it. Let a large sphere be surrounded by 30 small, identical spheres, each of which touches its four small-sphere neighbors as well as the large sphere. How is the radius of the large sphere related to that of the small spheres? (The inset shows a three-dimensional view of the problem.)



BRYAN CHRISTIE

**Answers are on page 91.*

creed the official closing of Japan. During this time of *sakoku,* or national seclusion, the government banned foreign books and travel, persecuted Christians and forbade Portuguese and Spanish ships from coming ashore. Many of these strictures would remain for more than two centuries, until Commodore Matthew C. Perry, backed by a fleet of U.S. warships, forced the end of *sakoku* in 1854.

Yet the isolationist policy was not entirely negative. Indeed, during the late 17th century, Japanese art and culture flowered so brilliantly that those years go by the name of *Genroku,* for "renaissance." In that era, *haiku* developed into a fine art form; No and Kabuki theater reached the pinnacle of their development; *ukiyo-e,* or "floating world" pictures, originated; and tea ceremonies and flower arranging reached new heights. Neither was mathematics left behind, for *Genroku* was also the age of Kowa Seki.

By popular accounts, Seki (1642–1708) was Japan's Isaac Newton or Gottfried Wilhelm Leibniz, although this reputation is difficult to substantiate. If the numbers of manuscripts attributed to him are correct, then most of his work has been lost. Still, there is no question that Seki left many disciples who were influential in the further development of Japanese mathematics.

The first—and incontestable—achievement of Seki was his theory of determinants, which is more powerful than that of Leibniz and which antedates the German mathematician's work by at least a decade. Another accomplishment, more relevant to temple geometry but of debatable origin, is the development of methods for solving high-degree equations. (Much traditional Japanese mathematics from that era involves equations to hundreds of degrees; one such equation is of the 1,458th degree.) Yet a third accomplishment sometimes attributed to Seki, and one that might also bear on *sangaku,* is the development of the *enri,* or circle principle.

The *enri* was quite similar to the method of exhaustion developed by Eudoxus and Archimedes in ancient Greece for computing the area of circles. The main difference was that Eudoxus and Archimedes used n-sided polygons to approximate the circle, whereas the *enri* divided the circle into n rectangles. Thus, the limiting procedure was somewhat different. Nevertheless, the *enri* represented a crude form of integral calculus that was later extended to other figures, including spheres and ellipses. A type of differential calculus was also developed around the same period. It is conceivable that the *enri* and similar techniques were brought to bear on *sangaku.* Today's mathematicians would use modern calculus to solve these problems.

Spheres within Ellipsoids

During Seki's lifetime, the first books employing the *enri* were published, and the first *sangaku* evidently made their appearance. The dates are almost certainly not coincidental; the followers of Yoshida and Seki must have influenced the development of *wasan,* and, in turn, *wasan* may have influenced them. Fukagawa believes that Seki encountered *sangaku* on his way to the shogunate castle, where he was officially employed as court mathematician, and that the tablets pushed him to further researches. A legend? Perhaps. But by the next century, books were being published that contained typical native Japanese problems: circles within triangles, spheres within pyramids, ellipsoids surrounding spheres. The problems found in these books do not differ in any important way from those found on the tablets, and it is difficult to avoid the conclusion that the peculiar flavor of all *wasan* problems—including the *sangaku*—is a direct result of the policy of national seclusion.

But the question immediately arises: Was Japan's isolation complete? It is certain that apart from the Dutch who were allowed to remain in Nagasaki Harbor on Kyushu, the southernmost island, all Western traders were banned. Equally clear is that the Japanese themselves were severely restricted. The mere act of traveling abroad was considered high treason, punishable by death. It appears safe to assume that if the isolation was not complete, then it was most nearly so, and any foreign influence on Japanese mathematics would have been minimal.

The situation began to change in the 19th century, when the *wasan* gradually became supplanted by *yosan,* a process that produced hybrid manuscripts written in *Kambun* with Western mathematical notations. And, after the opening of Japan by Commodore Perry and the subsequent collapse of the Tokugawa shogunate in 1867, the new government abandoned the study of native mathematics in favor of *yosan.* Some practitioners, however, continued to hang tablets well into the 20th century. A few *sangaku* even date from the current decade. But almost all the problems from this century are plagiarisms.

The final and most intriguing question is, Who produced the *sangaku?* Were the theorems so beautifully drawn on wooden tablets the works of professional mathematicians or amateurs? The evidence is meager.

Only a handful of *sangaku* are mentioned in the standard *A History of Japanese Mathematics,* by David E. Smith and Yoshio Mikami. They cite the 1789 collection *Shimpeki Sampo,* or *Mathematical Problems Suspended before the Temple,* which was published by Kagen Fujita, a professional mathematician. Smith and Mikami mention a tablet on which the following was appended after the solution: "Feudal district of Kakegawa in Enshu Province, third month of 1795, Sonobei Keichi Miyajima, pupil of Sadasuke Fujita of the School of Seki." Mikami, in his *Development of Mathematics in China and Japan,* mentions the "Gion Temple Problem," which was suspended at the Gion Temple in Kyoto by Enkyu Tsuda, pupil of Enri Nishimura. Furthermore, the tablets were written in the specialized language of *Kambun,* signifying the mark of an educated class of practitioners.

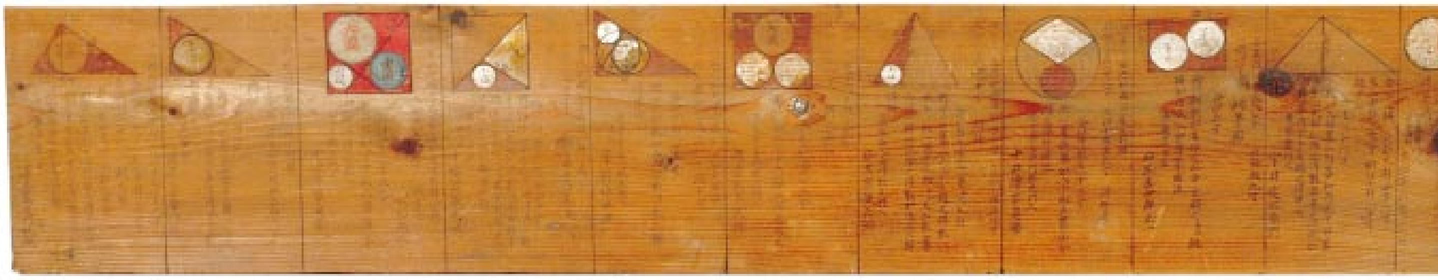
From such scraps of information, it is tempting to conclude that the tablets were the work primarily of professional mathematicians and their students. Yet there are reasons to believe otherwise.

Many of the problems are elementary and can be solved in a few lines; they are not the kind of work a professional mathematician would publish. Fukagawa has found a tablet from Mie Prefecture inscribed with the name of a merchant. Others have names of women and children—12 to 14 years of age. Most, according to Fukagawa, were created by the members of the highly educated samurai class. A few were probably done by farmers; Fukagawa recalls how about 10 years ago he visited the former cottage of mathematician Sen Sakuma (1819–1896), who taught *wasan* to the farmers in nearby villages in Fukushima Prefecture. Sakuma had about 2,000 students.

Such instruction recalls the Edo period itself, when there were no colleges or universities in Japan. During that time, teaching was carried out at private schools or temples, where ordinary people would go to study reading, writ-

Works of Art

Sangaku can be found in many Shinto shrines and Buddhist temples throughout Japan (*near right*). The tablets are traditionally hung below the eaves of the religious buildings, in a centuries-old practice of worshipers presenting wooden tablets as acts of homage to their guiding spirits (*center right*). The *sangaku* contain mathematical problems that almost always deal with geometry (*far right*). Many of the tablets are delicately colored (*bottom left*); some have been engraved in gold (*bottom right*). —*T.R.*



ing and the abacus. Because laypeople are more often drawn to problems of geometry than of algebra, it would not be surprising if the tablets were painted with such artistic care specifically to attract nonmathematicians.

The best answer, then, to the question of who created temple geometry seems to be: everybody. On learning of the *sangaku,* Fukagawa came to understand that, in those days, many of the Japanese loved and enjoyed math, as well as poetry and other art forms.

It is pleasant to realize that some *sangaku* were the works of ordinary mathematics devotees, carried away by the beauty of geometry. Perhaps a village teacher, after spending the day with students, or a samurai warrior, after sharpening his sword, would retire to his study, light an oil lamp and lose the world to an intricate problem involving spheres and ellipsoids. Perhaps he would spend days working on it in peaceful contemplation. After finally arriving at a solution, he might allow himself a short rest to savor the result of his hard labor. Convinced the proof was a worthy offering to his guiding spirits, he would have the theorem inscribed in wood, hang it in his local temple and begin to consider the next challenge. Visitors would notice the colorful tablet and admire its beauty. Many people would leave wondering how the author arrived at such a miraculous solution. Some might decide to give the problem a try or to study geometry so that the attempt could be made. A few might leave asking, "What if the problem were changed just so…."

Something for us all to consider. SA

The Author

TONY ROTHMAN received his Ph.D. in 1981 from the Center for Relativity at the University of Texas at Austin. He did postdoctoral work at Oxford, Moscow and Cape Town, and he has taught at Harvard University. Rothman has published six books, most recently *Instant Physics*. His next book is *Doubt and Certainty,* with E.C.G. Sudarshan, to be published this fall by Helix Books/Addison-Wesley. He has also recently written a novel about nuclear fusion. SCIENTIFIC AMERICAN wishes to acknowledge the help of Hidetoshi Fukagawa in preparing this manuscript. Fukagawa received a Ph.D. in mathematics from the Bulgarian Academy of Science. He is a high school teacher in Aichi Prefecture, Japan.

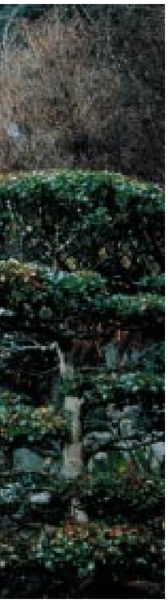
Further Reading

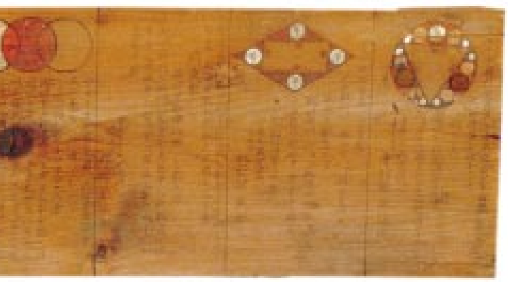
A HISTORY OF JAPANESE MATHEMATICS. David E. Smith and Yoshio Mikami. Open Court Publishing Company, Chicago, 1914. (Also available on microfilm.)

THE DEVELOPMENT OF MATHEMATICS IN CHINA AND JAPAN. Second edition (reprint). Yoshio Mikami. Chelsea Publishing Company, New York, 1974.

JAPANESE TEMPLE GEOMETRY PROBLEMS. H. Fukagawa and D. Pedoe. Charles Babbage Research Foundation, Winnipeg, Canada, 1989.

TRADITIONAL JAPANESE MATHEMATICS PROBLEMS FROM THE 18TH AND 19TH CENTURIES. H. Fukagawa and D. Sokolowsky. Science Culture Technology Publishing, Singapore (in press).

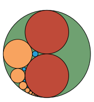


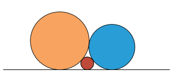

PHOTOGRAPHS BY HIROSHI UMEOKA

Answers to *Sangaku* Problems

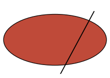
Unfortunately, because of space limitations the complete solutions to the problems could not be given here. Additional details can be found at http://www.sciam.com on the Scientific American *World Wide Web site.*



Answer: $r/[(2n-1)^2+14]$. The original solution to this problem applies the Japanese version of the Descartes circle theorem several times. The answer given here was obtained by using the inversion method, which was unknown to the Japanese mathematicians of that era.

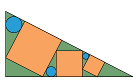


Answer: $1/\sqrt{r_3} = 1/\sqrt{r_1} + 1/\sqrt{r_2}$, where r_1, r_2 and r_3 are the radii of the orange, blue and red circles, respectively. The problem can be solved by applying the Pythagorean theorem.

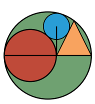


Answer: $PQ = \dfrac{\sqrt{27}\,a^2b^2}{(a^2+b^2)^{3/2}}$

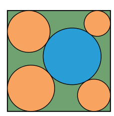
The problem can be solved by using analytic geometry to derive an equation for PQ and then taking the first derivative of the equation and setting it to zero to obtain the minimum value for PQ. It is not known whether the original authors resorted to calculus to solve this problem.



Answer: $r_2^{\,2} = r_1r_3$, where r_1, r_2 and r_3 are the radii of the large, medium and small blue circles, respectively. (In other words, r_2 is the geometric mean of r_1 and r_3.) The problem can be solved by first realizing that all the interior green triangles formed by the orange squares are similar. The original solution then looks at how the three squares are related.

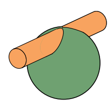


Answer: In the original solution to this problem, the author draws a line segment that goes through the center of the blue circle and is perpendicular to the drawn diameter of the green circle. The author assumes that this line segment is different from the line segment described in the statement of the problem on page 87. Thus, the two line segments should intersect the drawn diameter at different locations. The author then shows that the distance between those locations must necessarily be equal to zero—that is, that the two line segments are identical, thereby proving the perpendicularity.

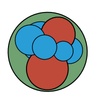


Answer: If a is the length of the square's side, and r_1, r_2, r_3 and r_4 are the radii of the upper right, upper left, lower left and lower right orange circles, respectively, then

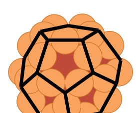
$$a = \frac{2(r_1r_3 - r_2r_4) + \sqrt{2(r_1-r_2)(r_1-r_4)(r_3-r_2)(r_3-r_4)}}{r_1 - r_2 + r_3 - r_4}$$



Answer: $16t\sqrt{t(r-t)}$, where r and t are the radii of the sphere and cylinder, respectively.



Answer: Six spheres. The Soddy hexlet theorem states that there must be six and only six blue spheres (thus the word "hexlet"). Interestingly, the theorem is true regardless of the position of the first blue sphere around the neck. Another intriguing result is that the radii of the different blue spheres in the "necklace" (t_1 through t_6) are related by $1/t_1 + 1/t_4 = 1/t_2 + 1/t_5 = 1/t_3 + 1/t_6$.



Answer: $R = \sqrt{5}r$, where R and r are the radii of the large and small spheres, respectively. The problem can be solved by realizing that the center of each small sphere lies on the midpoint of the edge of a regular dodecahedron, a 12-sided solid with pentagonal faces.

BRYAN CHRISTIE